Sea $\boldsymbol{L = \frac{1}{2}[\partial_\mu \phi\, \partial^\mu \phi - m^2 \phi]}$

a) Demostrar que es invariante bajo las transformadas de Lorentz

$$\boldsymbol{x^{0\prime} = \gamma(x^0 - \beta x^1)}$$
$$\boldsymbol{x^{1\prime} = \gamma(x^1 - \beta x^0)}$$
$$\boldsymbol{x^{2\prime} = x^2}$$
$$\boldsymbol{x^{3\prime} = x^3}$$

$$\beta = \frac{v}{c} \ ; \ \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

b) Calcular $\frac{\delta S[\phi]}{\delta \phi}$

Apartado a) En primer lugar, deshago la notación compacta, y pongo la expresión de L:

$$L = \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x^0}\right)^2 - \left(\frac{\partial \phi}{\partial x^1}\right)^2 - \left(\frac{\partial \phi}{\partial x^2}\right)^2 - \left(\frac{\partial \phi}{\partial x^3}\right)^2 - m^2 \phi\right]$$

Calcularé esas derivadas, pero suponiendo que $\phi(x^{0\prime}, x^{1\prime}, x^{2\prime}, x^{3\prime})$ está expresada en función de unas nuevas coordenadas (con prima), que se relacionan con las antiguas (sin prima) mediante las transformaciones de Lorentz puestas arriba. Utilizaré la regla de la cadena:

$$\frac{\partial \phi}{\partial x^0} = \frac{\partial \phi}{\partial x^{0\prime}}\frac{\partial x^{0\prime}}{\partial x^0} + \frac{\partial \phi}{\partial x^{1\prime}}\frac{\partial x^{1\prime}}{\partial x^0} + \frac{\partial \phi}{\partial x^{2\prime}}\frac{\partial x^{2\prime}}{\partial x^0} + \frac{\partial \phi}{\partial x^{3\prime}}\frac{\partial x^{3\prime}}{\partial x^0}$$

Según las T.de Lorentz: $\frac{\partial x^{0\prime}}{\partial x^0} = \gamma$; $\frac{\partial x^{1\prime}}{\partial x^0} = -\gamma\beta$; $\frac{\partial x^{2\prime}}{\partial x^0} = 0$; $\frac{\partial x^{3\prime}}{\partial x^0} = 0$. Por lo tanto:

$$\frac{\partial \phi}{\partial x^0} = \frac{\partial \phi}{\partial x^{0\prime}}(\gamma) + \frac{\partial \phi}{\partial x^{1\prime}}(-\gamma\beta) = \gamma\left[\frac{\partial \phi}{\partial x^{0\prime}} - \beta\frac{\partial \phi}{\partial x^{1\prime}}\right] \implies \boldsymbol{\left(\frac{\partial \phi}{\partial x^0}\right)^2 = \gamma^2\left(\frac{\partial \phi}{\partial x^{0\prime}}\right)^2 + \gamma^2\beta^2\left(\frac{\partial \phi}{\partial x^{1\prime}}\right)^2 - 2\gamma\beta\frac{\partial \phi}{\partial x^{0\prime}}\frac{\partial \phi}{\partial x^{1\prime}}}$$

De forma similar:

$$\frac{\partial \phi}{\partial x^1} = \frac{\partial \phi}{\partial x^{0\prime}}\frac{\partial x^{0\prime}}{\partial x^1} + \frac{\partial \phi}{\partial x^{1\prime}}\frac{\partial x^{1\prime}}{\partial x^1} + \frac{\partial \phi}{\partial x^{2\prime}}\frac{\partial x^{2\prime}}{\partial x^1} + \frac{\partial \phi}{\partial x^{3\prime}}\frac{\partial x^{3\prime}}{\partial x^1}$$

Según las T. de Lorentz: $\frac{\partial x^{0\prime}}{\partial x^1} = -\gamma\beta$; $\frac{\partial x^{1\prime}}{\partial x^1} = \gamma$; $\frac{\partial x^{2\prime}}{\partial x^1} = 0$; $\frac{\partial x^{3\prime}}{\partial x^1} = 0$. Por lo tanto:

$$\frac{\partial \phi}{\partial x^1} = \frac{\partial \phi}{\partial x^{0\prime}}(-\gamma\beta) + \frac{\partial \phi}{\partial x^{1\prime}}(\gamma) = \gamma\left(-\beta\frac{\partial \phi}{\partial x^{0\prime}} + \frac{\partial \phi}{\partial x^{1\prime}}\right) \implies \boldsymbol{\left(\frac{\partial \phi}{\partial x^1}\right)^2 = \gamma^2\beta^2\left(\frac{\partial \phi}{\partial x^{0\prime}}\right)^2 + \gamma^2\left(\frac{\partial \phi}{\partial x^{1\prime}}\right)^2 - 2\gamma\beta\frac{\partial \phi}{\partial x^{0\prime}}\frac{\partial \phi}{\partial x^{1\prime}}}$$

Para los casos de x^2 y x^3, al ser iguales a $x^{2\prime}$ y $x^{3\prime}$, respectivamente, se cumplirá que: $\frac{\partial \phi}{\partial x^2} = \frac{\partial \phi}{\partial x^{2\prime}}$ y $\frac{\partial \phi}{\partial x^3} = \frac{\partial \phi}{\partial x^{3\prime}}$
De todas formas, lo compruebo siguiendo la misma sistemática:

$$\frac{\partial \phi}{\partial x^2} = \frac{\partial \phi}{\partial x^{0\prime}}\frac{\partial x^{0\prime}}{\partial x^2} + \frac{\partial \phi}{\partial x^{1\prime}}\frac{\partial x^{1\prime}}{\partial x^2} + \frac{\partial \phi}{\partial x^{2\prime}}\frac{\partial x^{2\prime}}{\partial x^2} + \frac{\partial \phi}{\partial x^{3\prime}}\frac{\partial x^{3\prime}}{\partial x^2}$$

Según las T. de Lorentz: $\frac{\partial x^{0\prime}}{\partial x^2} = 0$; $\frac{\partial x^{1\prime}}{\partial x^2} = 0$; $\frac{\partial x^{2\prime}}{\partial x^2} = 1$; $\frac{\partial x^{3\prime}}{\partial x^2}$. Por lo tanto:

$$\frac{\partial \phi}{\partial x^2} = \frac{\partial \phi}{\partial x^{2\prime}} \implies \boldsymbol{\left(\frac{\partial \phi}{\partial x^2}\right)^2 = \left(\frac{\partial \phi}{\partial x^{2\prime}}\right)^2}$$

Igual la derivada respecto a x^3

$$\frac{\partial \phi}{\partial x^3} = \frac{\partial \phi}{\partial x^{0\prime}}\frac{\partial x^{0\prime}}{\partial x^3} + \frac{\partial \phi}{\partial x^{1\prime}}\frac{\partial x^{1\prime}}{\partial x^3} + \frac{\partial \phi}{\partial x^{2\prime}}\frac{\partial x^{2\prime}}{\partial x^3} + \frac{\partial \phi}{\partial x^{3\prime}}\frac{\partial x^{3\prime}}{\partial x^3}$$

Según las T. de Lorentz: $\frac{\partial x^{0\prime}}{\partial x^3} = 0$; $\frac{\partial x^{1\prime}}{\partial x^3} = 0$; $\frac{\partial x^{2\prime}}{\partial x^3} = 0$; $\frac{\partial x^{3\prime}}{\partial x^3} = 1$. Por lo tanto:

$$\frac{\partial \phi}{\partial x^3} = \frac{\partial \phi}{\partial x^{3\prime}} \implies \boldsymbol{\left(\frac{\partial \phi}{\partial x^3}\right)^2 = \left(\frac{\partial \phi}{\partial x^{3\prime}}\right)^2}$$

Tomando los resultados anteriores de las derivadas al cuadrado, que he destacado en negrita, expreso *L* en función de las nuevas coordenadas:

$$L = \frac{1}{2}\left[\gamma^2\left(\frac{\partial\phi}{\partial x^{0'}}\right)^2 + \gamma^2\beta^2\left(\frac{\partial\phi}{\partial x^{1'}}\right)^2 - 2\gamma\beta\frac{\partial\phi}{\partial x^{0'}}\frac{\partial\phi}{\partial x^{1'}} - \gamma^2\beta^2\left(\frac{\partial\phi}{\partial x^{0'}}\right)^2 - \gamma^2\left(\frac{\partial\phi}{\partial x^{1'}}\right)^2 + 2\gamma\beta\frac{\partial\phi}{\partial x^{0'}}\frac{\partial\phi}{\partial x^{1'}} - \left(\frac{\partial\phi}{\partial x^{2'}}\right)^2 - \left(\frac{\partial\phi}{\partial x^{3'}}\right)^2 - m^2\phi\right]$$

Se puede simplificar (el término $2\gamma\beta\frac{\partial\phi}{\partial x^{0'}}\frac{\partial\phi}{\partial x^{1'}}$ se va) y lo reescribo todo utilizando para las derivadas la notación relativista:

$$L = \frac{1}{2}[\gamma^2(\partial_{0'}\phi)^2 + \gamma^2\beta^2(\partial_{1'}\phi)^2 - \gamma^2\beta^2(\partial_{0'}\phi)^2 - \gamma^2(\partial_{1'}\phi)^2 - (\partial_{2'}\phi)^2 - (\partial_{3'}\phi)^2 - m^2\phi]$$

Sacando factores comunes:

$$L = \frac{1}{2}[\gamma^2(1-\beta^2)(\partial_{0'}\phi)^2 - \gamma^2(1-\beta^2)(\partial_{1'}\phi)^2 - (\partial_{2'}\phi)^2 - (\partial_{3'}\phi)^2 - m^2\phi]$$

Si tenemos en cuenta que: $\gamma^2(1-\beta^2) = \frac{1}{1-\beta^2}(1-\beta^2) = 1$ se llega a:

$$L = \frac{1}{2}[(\partial_{0'}\phi)^2 - (\partial_{1'}\phi)^2 - (\partial_{2'}\phi)^2 - (\partial_{3'}\phi)^2 - m^2\phi] = \boldsymbol{\frac{1}{2}[(\partial_{\mu'}\phi)(\partial^{\mu'}\phi) - m^2\phi]}$$

Comprobando así que la expresión de L tiene la misma forma, expresada con coordenadas $\{x^\mu\}$ o con $\{x^{\mu'}\}$

<u>Apartado b</u>) Cuando la función ϕ de la que depende el funcional $S[\phi]$ es de varias variables $\phi(x^0, x^1, x^2, x^3)$ la derivada del funcional es:

$$\frac{\delta S[\phi]}{\delta\phi} = \frac{\partial L}{\partial\phi} - \frac{\partial}{\partial x^0}\left(\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^0}}\right) - \frac{\partial}{\partial x^1}\left(\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^1}}\right) - \frac{\partial}{\partial x^2}\left(\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^2}}\right) - \frac{\partial}{\partial x^3}\left(\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^3}}\right)$$

$$L = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x^0}\right)^2 - \left(\frac{\partial\phi}{\partial x^1}\right)^2 - \left(\frac{\partial\phi}{\partial x^2}\right)^2 - \left(\frac{\partial\phi}{\partial x^3}\right)^2 - m^2\phi\right]$$

Con la expresión de L hacemos las derivadas:

$$\frac{\partial L}{\partial\phi} = -m\phi$$

$$\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^0}} = \frac{\partial\phi}{\partial x^0} \quad\rightarrow\quad \frac{\partial}{\partial x^0}\left(\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^0}}\right) = \frac{\partial^2\phi}{\partial {x^0}^2} = \partial_0^2\phi$$

$$\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^1}} = -\frac{\partial\phi}{\partial x^1} \quad\rightarrow\quad \frac{\partial}{\partial x^1}\left(\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^1}}\right) = -\frac{\partial^2\phi}{\partial {x^1}^2} = -\partial_1^2\phi$$

$$\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^2}} = -\frac{\partial\phi}{\partial x^2} \quad\rightarrow\quad \frac{\partial}{\partial x^2}\left(\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^2}}\right) = -\frac{\partial^2\phi}{\partial {x^2}^2} = -\partial_2^2\phi$$

$$\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^3}} = -\frac{\partial\phi}{\partial x^3} \quad\rightarrow\quad \frac{\partial}{\partial x^3}\left(\frac{\partial L}{\partial\frac{\partial\phi}{\partial x^3}}\right) = -\frac{\partial^2\phi}{\partial {x^3}^2} = -\partial_3^2\phi$$

Sustituyendo en la expresión de la derivada, queda:

$$\boldsymbol{\frac{\delta S[\phi]}{\delta\phi} = -m\phi - \partial_0^2\phi + \partial_1^2\phi + \partial_2^2\phi + \partial_3^2\phi}$$